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# Broué's conjecture for the nonprincipal block of $SL(2,q)$ with full defect

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ABSTRACT

M. Broué gives an important conjecture which is called *Broué's abelian defect group conjecture*. This conjecture says that a $p$-block, where $p$ is a prime number, of a finite group with an abelian defect group is derived equivalent to its Brauer correspondent in the normalizer of the defect group. In this paper, we prove that this conjecture is true for the nonprincipal block of $SL(2,p^n)$ for a positive integer $n$.

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## 1. Introduction

*1.1.* On the block theory of finite groups, the following conjecture is one of the most important problems:

**Broué's abelian defect group conjecture.** *(See [2, 6.2.Question].) Let $k$ be an algebraically closed field of characteristic $p>0$, and let $G$ be a finite group. If $A$ is a block of $kG$ with abelian defect group $P$ and $B$ is the Brauer correspondent of $A$ in $kN_G(P)$, then is $A$ derived equivalent to $B$?*

The conjecture has been studied by many people including Broué, Okuyama, Rickard, Rouquier, Linckelmann, Puig, Chuang, Kessar, Hida, Miyachi, Waki, Kunugi, Koshitani, and so on (see [5, §9]). If $G=SL(2,q)$ where $q=p^n$, it has been proved that the conjecture is true for the principal block by Rouquier in [14] for $p^n=8$, by Chuang in [4] for $n=2$, and by Okuyama in [10] for the general case. In the nonprincipal block case, Holloway proved it in [7] for $n=2$, but it has not been known for $n \geqslant 3$ yet. However, it has turned out that even in the nonprincipal block case, we can prove that the conjecture is true:

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**Theorem 1.1.1.** *Broué's abelian defect group conjecture is true for the nonprincipal block* (*with full defect*) *of* $kSL(2,q)$.

Our method is essentially similar to Okuyama's for the principal block case.

In Section 2, we describe general theory including how to construct tilting complexes which is used to prove the theorem. These arguments are followed by Okuyama [10], but the proofs are omitted.

In Section 3, we describe representation theory of $kSL(2,q)$ and its nonprincipal block. As in Section 2, we follow Okuyama [10, §2] until the end of 3.2. The key to prove the theorem is in 3.3. In the principal block case, Okuyama orders some "equivalence classes" of the set of all nonisomorphic simple modules in the principal block (see [10, §2]). In fact, even in the nonprincipal block case, there exist such ordered "equivalence classes" as enable us to apply Okuyama's method.

In Section 4, we prove the theorem. The proof is similar to that in Okuyama [10, §3], but for the nonprincipal block $A$ of $kSL(2,q)$ and the Brauer correspondent $B$ in $kN_G(P)$, we use the composition factors of projective indecomposable modules of $kSL(2,q)$ to prove the part (f) in Proposition 4.1.1 because they are completely known (see e.g. [1,6]). Moreover, we also remark that the result extends to $kGL(2,q)$.

*1.2.* Here we shall introduce some notations. If $k$ is an algebraically closed field and if $A$ is a finite dimensional $k$-algebra with unit $1_A$, then let $A^{op}$ be the opposite algebra of $A$, let mod-$A$ be the category consisting of all finite dimensional right $A$-modules, and let $K^b(\text{mod-}A)$ be the homotopy category consisting of all bounded complexes of finite dimensional right $A$-modules. If $M$ is a right $A$-module, $\mathrm{rad}(M)$ denotes its radical, $\mathrm{soc}(M)$ denotes its socle, $\mathrm{top}(M)$ denotes $M/\mathrm{rad}(M)$ and $\mathrm{heart}(M)$ denotes $\mathrm{rad}(M)/\mathrm{soc}(M)$ if it exists. If $V$ is a $k$-vector space, $V^*$ denotes the dual $k$-vector space $\mathrm{Hom}_k(V,k)$, and $\otimes$ denotes $\otimes_k$. Moreover, if $H$ is a subgroup of a finite group $G$, then $M\uparrow^G$ and $N\downarrow_H$ denote the induction and the restriction for a $kH$-module $M$ and a $kG$-module $N$, respectively.

## 2. General theory

We shall give materials which are necessary to prove the main theorem following Okuyama [10] (but without proof).

*2.1.* Let $k$ be an algebraically closed field of characteristic $p>0$, let $A$ and $B$ be finite dimensional $k$-algebras (with units $1_A$ and $1_B$, respectively), and let ${}_BM_A$ be a $(B,A)$-bimodule ($=$ right $(B^{op}\otimes_k A)$-module) inducing a stable equivalence of Morita type. Suppose that $A$ and $B$ are indecomposable $k$-algebras, and that ${}_BM_A$ has no nonzero projective summands. Then ${}_BM_A$ is an indecomposable $(B,A)$-bimodule.

Let $T_i$ $(i\in I)$ be all nonisomorphic (right) $B$-modules, $\tau_i : Q_i \to T_i$ the projective cover of $T_i$, and $\pi_i : P_i \to T_i\otimes_B M$ the projective cover of $(T_i\otimes_B M)_A$. Then there exists an $A$-homomorphism $\rho_i : P_i \to Q_i\otimes_B M$ such that $\pi_i = (\tau_i\otimes id_M)\circ\rho_i$ by the projectivity of $P_i$ and $\rho_i$ corresponds to an $A$-homomorphism $\delta_i : Q_i^*\otimes P_i \to M$ through the natural isomorphism $\mathrm{Hom}_A(P_i, Q_i\otimes_B M)\cong \mathrm{Hom}_{B^{op}\otimes A}(Q_i^*\otimes P_i, M)$. Then, $\bigoplus_{i\in I}\delta_i : \bigoplus_{i\in I} Q_i^*\otimes P_i \to M$ is the projective cover of $(B,A)$-bimodule $M$, and by applying the functor $T_i\otimes_B -$, we obtain the projective cover $\pi_i : P_i \to T_i\otimes_B M$.

Let $I_0$ be a fixed subset of $I$, and define the bounded complex $M(I_0)^\bullet$ of $(B,A)$-bimodules as

$$\cdots 0 \to \bigoplus_{i\in I_0} Q_i^*\otimes P_i \xrightarrow{\bigoplus_{i\in I_0}\delta_i} M \to 0\cdots.$$

(The construction of this complex is based on Rouquier [13].) We shall consider the following condition:

**Condition 2.1.1.** *For any $i \in I_0$ and $j \in I - I_0$,*

(a) $\mathrm{Hom}_A(T_j \otimes_B M, \mathrm{Ker}\,\pi_i) = 0$.
(b) *Any $A$-homomorphism $P_i \to T_j \otimes_B M$ factors through $\pi_i$.*

This condition is a criterion for $M(I_0)^\bullet$ being a tilting complex for $A$ (cf. [11, §6]):

**Theorem 2.1.1.** *The following are equivalent.*

(a) *$M(I_0)^\bullet$ is a tilting complex for $A$ (not for $B^{op} \otimes A$).*
(b) *$M$ satisfies Condition* 2.1.1.

*2.2.* Throughout this subsection, suppose that Condition 2.1.1 holds, namely, $M(I_0)^\bullet$ is a tilting complex for $A$.

Set $C = \mathrm{End}_{K^b(\text{mod-}A)}(M(I_0)^\bullet)$. Then $C$ is derived equivalent to $A$ by the assumption.

**Proposition 2.2.1.**

(a) *There exists a (unitary) $k$-algebra monomorphism from $B$ to $C$.*
(b) *${}_BC_C$ induces a stable equivalence of Morita type between $B$ and $C$, and ${}_CC_B^* \cong {}_CC_B$.*
(c) *${}_BC_C$ is projective free (hence indecomposable).*

Since ${}_CC_B$ induces a stable equivalence of Morita type between $B$ and $C$, and ${}_BM_A$ induces a stable equivalence of Morita type between $A$ and $B$, we see that ${}_C(C \otimes_B M)_A$ induces a stable equivalence of Morita type between $A$ and $C$. So ${}_C(C \otimes_B M)_A \cong N \oplus (\text{proj.}(C, A)\text{-bimodule})$, where $N$ is a nonprojective and indecomposable $(C, A)$-bimodule.

**Proposition 2.2.2.** *Let $N$ be a nonprojective indecomposable summand of ${}_C(C \otimes_B M)_A$, and suppose that, for a simple $A$-module $S$,*

$$\mathrm{Hom}_A(S, T_i \otimes_B M) = 0 = \mathrm{Hom}_A(T_i \otimes_B M, S) \quad \text{for any } i \in I_0.$$

*Then $\mathrm{Hom}_A(N, S)(\cong S \otimes_A N^*)$ is a simple $C$-module.*

We shall consider the following condition:

**Condition 2.2.1.**

(a) *For $i \in I_0$, $\dim \mathrm{Hom}_A(\Omega(T_i \otimes_B M), \Omega(T_l \otimes_B M)) = \delta_{il}$ for any $l \in I_0$.*
(b) *For $j \notin I_0$, $\dim \mathrm{Hom}_A(T_j \otimes_B M, T_l \otimes_B M) = \delta_{jl}$ for any $l \notin I_0$.*

For $i \in I$, this condition is a criterion for $(T_i \otimes_B C)_C$ being simple:

**Proposition 2.2.3.**

(a) *If $i \in I_0$ satisfies Condition* 2.2.1(a), *then $(T_i \otimes_B C)_C$ is simple.*
(b) *If $j \notin I_0$ satisfies Condition* 2.2.1(b), *then $(T_i \otimes_B C)_C$ is simple.*

**Corollary 2.2.1.** *Suppose that any $i \in I_0$ satisfies Condition* 2.2.1(a), *and that any $j \notin I_0$ satisfies Condition* 2.2.1(b). *Then $M(I_0)^\bullet$ is a Rickard tilting complex for $(B, A)$, namely,*

$$M(I_0)^\bullet \overset{\bullet}{\otimes}_A M(I_0)^{\bullet *} \cong B, \quad \text{in } K^b(\text{mod-}B^{op} \otimes B)$$

*and*

$$M(I_0)^{\bullet *} \overset{\bullet}{\otimes}_B M(I_0)^{\bullet} \cong A, \quad \text{in } K^b\big(\text{mod-}A^{op} \otimes A\big),$$

*where* $M(I_0)^{\bullet} \overset{\bullet}{\otimes}_A M(I_0)^{\bullet *}$ *and* $M(I_0)^{\bullet *} \overset{\bullet}{\otimes}_B M(I_0)^{\bullet}$ *are the total complexes of the double complexes* $M(I_0)^{\bullet} \overset{\bullet\bullet}{\otimes}_A M(I_0)^{\bullet *}$ *and* $M(I_0)^{\bullet *} \overset{\bullet\bullet}{\otimes}_B M(I_0)^{\bullet}$, *respectively* (*see* [12]).

*2.3.* In this subsection we assume furthermore that there is a permutation "$\widetilde{\cdot}$" in $I$ of order 2. For a subset $I_0$ of $I$, set $J_0 = \widetilde{I_0}$ and $K_0 = I_0 \cup J_0$. We shall consider the following condition.

**Condition 2.3.1.** *There exists a set of simple $A$-modules $S_l$, $l \in K_0$ satisfying the following*:

(1) *For $i \in K_0$, $T_i \otimes_B M$ is nonsimple and*
  (i) $\operatorname{top}(T_i \otimes_B M) \cong S_i \cong \operatorname{soc}(T_{\widetilde{i}} \otimes_B M)$,
  (ii) $\operatorname{soc}(\operatorname{Hom}_A(M, S_i)) \cong T_i \cong \operatorname{top}(\operatorname{Hom}_A(M, S_{\widetilde{i}}))$.
(2) *If $P_i$ is a projective cover of $S_i$, then*
  (i) *for $i \in I_0$,*

$$\dim \operatorname{Hom}_A(P_i, P_i) = \begin{cases} 2 & (\textit{if } \widetilde{i} \neq i), \\ 3 & (\textit{if } \widetilde{i} = i), \end{cases}$$

  (ii) *for $i, l \in I_0$, $\operatorname{Hom}_A(P_i, P_l) = 0$ if $l \neq i$ and $l \neq \widetilde{i}$,*
  (iii) *for $i \in I_0$, $\dim \operatorname{Hom}_A(P_i, P_{\widetilde{i}}) = 1$ if $\widetilde{i} \neq i$.*
(3) *For $i \in I_0$, $\operatorname{Hom}_A(P_i, T_l \otimes_B M) = 0$ if $l \notin K_0$.*

**Remark.** If the Cartan matrix of $A$ is positive definite, the condition (2)(iii) automatically holds because of (1) and the rest of (2).

**Proposition 2.3.1.** *If $K_0$ satisfies Condition* 2.3.1, *then*

(a) *$I_0$ satisfies Condition* 2.1.1, *namely, $M(I_0)^{\bullet}$ is a tilting complex for $A$.*
(b) (i) *Any $i \in I_0$ satisfies Condition* 2.2.1(a).
  (ii) *Any $j \in K_0 - I_0$ satisfies Condition* 2.2.1(b).

## 3. Representation theory of $SL(2, q)$

Set $G = SL(2, q)$ where $q = p^n$ for a positive integer $n$, and let $k$ be an algebraically closed field of characteristic $p$. In this section we shall state facts about representations of $kG$ following Okuyama [10] until the end of 3.2 and about its nonprincipal block (so assume $p \neq 2$ in the rest of the paper). Set

$$P = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{F}_q \right\}, \qquad D = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{F}_q^{\times} \right\},$$

and

$$H = N_G(P) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{F}_q^{\times},\ b \in \mathbb{F}_q \right\},$$

where $\mathbb{F}_q$ is the finite field of $q$ elements. Then $P$ is a Sylow $p$-subgroup of $G$ and hence is isomorphic to the elementary abelian group $C_p \times \cdots \times C_p$ ($n$ times), $D$ is isomorphic to $C_{q-1}$, where $C_r$ is the cyclic group of order $r$, and $H$ is a semidirect product $P \rtimes D$.

*3.1.* For an integer $\lambda$, let $T_\lambda$ be a one-dimensional space. Then $T_\lambda$ is a (right) $kH$-module by

$$v\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a^\lambda v \quad \text{for } v \in T_\lambda \quad \text{and} \quad \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in H.$$

Note that $T_0$ is the trivial $kH$-module and $T_\lambda \cong T_{\lambda+q-1}$, so $T_\lambda$ are all nonisomorphic $kH$-modules for $\lambda = 0, 1, \dots, q-1$.

For a $kG$-module $M$, let $M^{(i)}$ be the twist of $M$ by the $i$th Frobenius map

$$F : G \to G, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^{p^i} & b^{p^i} \\ c^{p^i} & d^{p^i} \end{pmatrix}.$$

Let $E$ be the natural (right) $kG$-module, and for $\mu = 0, 1, \dots, p-1$, let $S_\mu = \mathrm{Sym}^\mu(E)$ be the $\mu$th symmetric power of $E$. Then for $\lambda = 0, 1, \dots, q-1$ and its $p$-adic expansion $\lambda = \sum_{i=0}^{n-1} \lambda_i p^i$, it is well known that

$$S_\lambda = S_{\lambda_0}^{(0)} \otimes S_{\lambda_1}^{(1)} \otimes \cdots \otimes S_{\lambda_{n-1}}^{(n-1)}$$

is a simple $kG$-module and all simple $kG$-modules are of these forms, so the simple $kG$-modules are indexed by $\{0, 1, \dots, q-1\}$, see [6]. Note that $S_0$ is the trivial $kG$-module and $S_{q-1}$ is projective.

*3.2.* Set $\Lambda = \{0, 1, \dots, q-1\}$ and $\Lambda_0 = \Lambda - \{q-1\}$, and for $\lambda \in \Lambda$ and its $p$-adic expansion $\lambda = \sum_{i=0}^{n-1} \lambda_i p^i$, let $V(\lambda)$ be the subset of $\Lambda$ consisting of all $\mu \in \Lambda$ satisfying $\mu \equiv \sum_{i=0}^{n-1} \varepsilon_i \lambda_i p^i \pmod{q-1}$ for some $\varepsilon_i \in \{1, -1\}$ $(0 \leqslant i \leqslant n-1)$, and let $W(\lambda)$ be the subset of $\Lambda$ consisting of all $\mu \in \Lambda$ satisfying $\mu = \sum_{i=0}^{n-1} \varepsilon'_i \lambda_i p^i$ for some $\varepsilon'_i \in \{1, -1\}$ $(0 \leqslant i \leqslant n-1)$.

We shall define a permutation and two relations in $\Lambda_0$. First, we define the permutation "$\widetilde{\cdot}$" of order 2 by

$$\widetilde{\lambda} = \begin{cases} 0 & \text{if } \lambda = 0, \\ q-1-\lambda & \text{if } \lambda \neq 0, \end{cases}$$

and the subset $\widetilde{\Omega} = \{\widetilde{\lambda} \mid \lambda \in \Omega\}$ for a subset $\Omega \subseteq \Lambda_0$, and next, the equivalence relation "$\sim$" by

$$\begin{aligned} \lambda \sim \mu \quad &\overset{\text{def}}{\Leftrightarrow} \quad \lambda \equiv p^j \mu \pmod{q-1} \quad \text{for some } j \in \{0, \dots, n-1\} \\ &\Leftrightarrow \quad S_\lambda \cong S_\mu^{(j)}, \end{aligned}$$

and finally, the partial order "$\preccurlyeq$" by

$$\text{for } \lambda,\ \lambda' \text{ and their } p\text{-adic expansions} \quad \lambda = \sum_{i=0}^{n-1} \lambda_i p^i, \qquad \lambda' = \sum_{i=0}^{n-1} \lambda'_i p^i,$$

$$\lambda \preccurlyeq \lambda' \quad \overset{\text{def}}{\Leftrightarrow} \quad \lambda_i \leqslant \lambda'_i \quad \text{and} \quad \lambda'_i - \lambda_i \text{ is even for each } i \text{ with } 0 \leqslant i \leqslant n-1.$$

It is proved that $V(\lambda) = W(\lambda) \cup \widetilde{W(\lambda)}$ for $\lambda \in \Lambda_0$, see [6], and note that $T_{\widetilde{\lambda}}$ is isomorphic to the dual $kH$-module $T_\lambda^*$ of $T_\lambda$. Then we can give some facts about inductions and restrictions:

- For $0 \leqslant \lambda \leqslant q-1$,
  (1) $\mathrm{soc}(S_\lambda \downarrow_H) \cong T_\lambda$ and $\mathrm{top}(S_\lambda \downarrow_H) \cong T_{\widetilde{\lambda}}$.
  (2) Every composition factor of $S_\lambda \downarrow_H$ is isomorphic to some $T_\mu$ with $\mu \in V(\lambda')$ and $\lambda' \preccurlyeq \lambda$.

- For $0 < \mu < q-1$,
  (3) $\mathrm{top}(T_\mu \uparrow^G) \cong S_\mu$ and $\mathrm{soc}(T_\mu \uparrow^G) \cong S_{\widetilde{\mu}}$.
  (4) $T_\mu \uparrow^G \leftrightarrow \bigoplus_{\lambda \in \mathcal{A}_\mu} S_\lambda$ if $\mu \neq (q-1)/2$, $T_\mu \uparrow^G \leftrightarrow 2(\bigoplus_{\lambda \in \mathcal{A}_\mu} S_\lambda)$ if $\mu = (q-1)/2$, where $\mathcal{A}_\mu = \{\lambda \in \Lambda_0 | \mu \in V(\widetilde{\lambda})\}$ and $M \leftrightarrow M'$ denotes that $M$ and $M'$ have the same composition factors with multiplicities.

**Lemma 3.2.1.** *Let $\lambda, \lambda' \in \Lambda_0$ satisfy $\lambda \sim \lambda'$, then*

(a) *If $\mu' \in V(\lambda')$, then there exists $\mu \in \Lambda_0$ such that $\mu' \sim \mu \in V(\lambda)$.*
(b) *If $\mu' \preccurlyeq \lambda'$, then there exists $\mu \in \Lambda_0$ such that $\mu' \sim \mu \preccurlyeq \lambda$.*

**Proof.** (a) Let $\lambda = \sum_{i=0}^{n-1} \lambda_i p^i$ and $\lambda' = \sum_{i=0}^{n-1} \lambda'_i p^i$ be $p$-adic expansions. Since $\lambda \sim \lambda'$, there is an integer $j \in \{0, \ldots, n-1\}$ with $\lambda' \equiv p^j \lambda \pmod{q-1}$, then we have $\sum_{i=0}^{n-1} \lambda'_i p^i \equiv \sum_{i=0}^{n-1} \lambda_i p^{i+j} \equiv \sum_{i=0}^{n-1} \lambda_{i-j} p^i \pmod{q-1}$, but we consider the indices $i$ in $\lambda_i$ as elements of $\mathbb{Z}/n\mathbb{Z}$, so

$$\lambda'_i = \lambda_{i-j} \quad \text{for all } i = 0, 1, \ldots, n-1. \tag{$*$}$$

Now $\mu' \in V(\lambda')$, we have $\mu' \equiv \sum_{i=0}^{n-1} \varepsilon_i \lambda'_i p^i = \sum_{i=0}^{n-1} \varepsilon_i \lambda_{i-j} p^i \pmod{q-1}$, for some $\varepsilon_i \in \{1, -1\}$, $0 \leqslant i \leqslant n-1$. By multiplying this by $p^{n-j}$, we have $p^{n-j}\mu' \equiv \sum_{i=0}^{n-1} \varepsilon_i \lambda_{i-j} p^{i+n-j} \equiv \sum_{i=0}^{n-1} \varepsilon_{i+j} \lambda_i p^i \pmod{q-1}$, but we also consider the indices $i$ in $\varepsilon_i$ as elements of $\mathbb{Z}/n\mathbb{Z}$. Now if we choose $\mu \in \Lambda_0$ satisfying $\mu \equiv p^{n-j}\mu' \pmod{q-1}$, then we have $\mu \in V(\lambda)$ and $\mu \sim \mu'$.

(b) Let $\mu' = \sum_{i=0}^{n-1} \mu'_i p^i$ be the $p$-adic expansion. If we choose $\mu = \sum_{i=0}^{n-1} \mu_i p^i$ satisfying $\mu_{i-j} = \mu'_i$ at $(*)$ in the proof of (a), then $\mu$ satisfies $\mu \preccurlyeq \lambda$ and $\mu \sim \mu'$. $\square$

**Remark.** In (a), if $\mu' \neq \lambda'$ (resp. $\widetilde{\lambda'}$), then $(\varepsilon_0, \ldots, \varepsilon_{n-1}) \neq (1, \ldots, 1)$ (resp. $(-1, \ldots, -1)$), so we can choose $\mu$ satisfying $\mu \neq \lambda$ (resp. $\widetilde{\lambda}$). Similarly, in (b), if $\mu' \neq \lambda'$, we can choose $\mu$ satisfying $\mu \neq \lambda$.

Set $I_{\text{even}} = \{0, 2, \ldots, q-3\}$ and $I_{\text{odd}} = \{1, 3, \ldots, q-2\}$. It is well known that the simple $kG$-modules in the principal block (resp. nonprincipal block with full defect) are indexed by $I_{\text{even}}$ (resp. $I_{\text{odd}}$).

*3.3.* In this subsection we shall study about the nonprincipal block. Set $I = I_{\text{odd}}$.

We define the ordered equivalence classes (with respect to "$\sim$") $J_{-1}, J_0, \ldots, J_s$ in $I$ as follows:

Let $J_{-1}$ and $\widetilde{J_{-1}}$ be the empty sets (by convention), let $J_0$ be the class containing 1, and $J_i$ the class containing the smallest $\lambda_i \notin \bigcup_{u=-1}^{i-1} (J_u \cup \widetilde{J_u})$ for $i \geqslant 1$. We repeat this procedure until $s$ satisfies $I = \bigcup_{u=-1}^{s} (J_u \cup \widetilde{J_u})$.

**Lemma 3.3.1.** *For each $t$ with $0 \leqslant t \leqslant s$ and any $\lambda \in J_t$,*

(a) $V(\lambda) - \{\lambda, \widetilde{\lambda}\} \subseteq \bigcup_{u=-1}^{t-1} (J_u \cup \widetilde{J_u})$.
(b) *If $\lambda' \precnsim \lambda$, then $V(\lambda') \subseteq \bigcup_{u=-1}^{t-1} (J_u \cup \widetilde{J_u})$.*

**Proof.** To begin with, we shall show in the case of $t = 0$. For any $\lambda \in J_0$, its $p$-adic expansion is of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$, so $W(\lambda) = \{\lambda\}$ and there is no $\lambda'$ such that $\lambda' \precnsim \lambda$ (so (b) follows). Then $V(\lambda) = W(\lambda) \cup \widetilde{W(\lambda)} = \{\lambda, \widetilde{\lambda}\}$ and (a) follows. So suppose that $t \geqslant 1$.

(a) Actually, it suffices to prove only for the smallest $\lambda_t \in J_t$. Indeed, if we had $V(\lambda_t) - \{\lambda_t, \widetilde{\lambda_t}\} \subseteq \bigcup_{u=-1}^{t-1} (J_u \cup \widetilde{J_u})$, then for any $\mu \in V(\lambda) - \{\lambda, \widetilde{\lambda}\}$, there would exist $\mu_t$ such that $\mu \sim \mu_t \in V(\lambda_t) - \{\lambda_t, \widetilde{\lambda_t}\}$ by Lemma 3.2.1(a) and its remark, and so $\mu \in \bigcup_{u=-1}^{t-1} (J_u \cup \widetilde{J_u})$, as desired. Moreover, since $V(\lambda_t) = W(\lambda_t) \cup \widetilde{W(\lambda_t)}$, it suffices to show that $W(\lambda_t) - \{\lambda_t\} \subseteq \bigcup_{u=-1}^{t-1} (J_u \cup \widetilde{J_u})$. For any $\mu \in W(\lambda_t) - \{\lambda_t\}$, $\mu$ is less than $\lambda_t$, so $\mu$ must be in $\bigcup_{u=-1}^{t-1} (J_u \cup \widetilde{J_u})$ by the smallness of $\lambda_t$.

(b) It also suffices to prove only for the smallest $\lambda_t \in J_t$. Indeed, suppose that $V(\lambda'_t) \subseteq \bigcup_{u=-1}^{t-1} (J_u \cup \widetilde{J_u})$ for any $\lambda'_t \precnsim \lambda_t$. Then for any $\lambda' \precnsim \lambda$, by Lemma 3.2.1(b) and its remark, there exists $\lambda'_t$ such that

$\lambda' \sim \lambda'_t \precneqq \lambda_t$. Then for any $\mu' \in V(\lambda')$, by Lemma 3.2.1(a), there exists $\mu'_t$ such that $\mu' \sim \mu'_t \in V(\lambda'_t)$. But since $V(\lambda'_t) \subseteq \bigcup_{u=-1}^{t-1}(J_u \cup \widetilde{J_u})$, $\mu'$ must be in $\bigcup_{u=-1}^{t-1}(J_u \cup \widetilde{J_u})$, so $V(\lambda') \subseteq \bigcup_{u=-1}^{t-1}(J_u \cup \widetilde{J_u})$, as desired. Moreover, since $V(\lambda'_t) = W(\lambda'_t) \cup \widetilde{W(\lambda'_t)}$, it suffices to show that $W(\lambda'_t) \subseteq \bigcup_{u=-1}^{t-1}(J_u \cup \widetilde{J_u})$. For any $\mu'_t \in W(\lambda'_t)$, we have $\mu'_t \leqslant \lambda'_t$ ($\precneqq \lambda_t$), so $\mu'_t$ must be in $\bigcup_{u=-1}^{t-1}(J_u \cup \widetilde{J_u})$ by the smallness of $\lambda_t$. $\square$

**Lemma 3.3.2.** *For any $\lambda \in J_t$ and any $\mu \in J_u \cup \widetilde{J_u}$ ($0 \leqslant u, t \leqslant s$, $\mu \neq \lambda$ and $\mu \neq \widetilde{\lambda}$),*

(a) *if $S_{\widetilde{\lambda}}$ is a composition factor of $T_\mu \uparrow^G$, then $u < t$,*
(b) *if $T_\mu$ is a composition factor of $S_\lambda \downarrow_H$, then $u < t$.*

**Proof.** (a) By 3.2(4), $S_{\widetilde{\lambda}}$ is a composition factor of $T_\mu \uparrow^G$ if and only if $\mu \in V(\lambda)$. Therefore, by Lemma 3.3.1(a), $\mu$ must lie in $\bigcup_{v=-1}^{t-1}(J_v \cup \widetilde{J_v})$, so $u < t$.

(b) If $T_\mu$ is a composition factor of $S_\lambda \downarrow_H$, there exists some $\lambda'$ such that $\lambda' \preccurlyeq \lambda$ and $\mu \in V(\lambda')$ by 3.2(2).

- Assume $\lambda' = \lambda$.
  Then $\mu$ lies in $V(\lambda) - \{\lambda, \widetilde{\lambda}\}$ by the assumptions $\mu \neq \lambda$ and $\mu \neq \widetilde{\lambda}$. By Lemma 3.3.1(a), $\mu$ must lie in $\bigcup_{v=-1}^{t-1}(J_v \cup \widetilde{J_v})$, so $u < t$.
- Assume $\lambda' \precneqq \lambda$.
  Then, by Lemma 3.3.1(b), $\mu \in V(\lambda') \subseteq \bigcup_{v=-1}^{t-1}(J_v \cup \widetilde{J_v})$, so $u < t$. $\square$

Let $A$ be the nonprincipal block of $kG$ with full defect and $B$ the Brauer correspondent of $A$ in $kH$. Set $I_t = \widetilde{J_t}$ and $K_t = I_t \cup J_t$, so $I = \bigcup_{u=-1}^{s} K_u$. Then $M = {}_BA_A$ induces a stable equivalence of Morita type since a Sylow $p$-subgroup of $G$ has trivial intersection. The dual $M^*$ of $M$ is isomorphic to ${}_AA_B$, so for any simple $A$-module $S_A$, we have $\mathrm{Hom}_A(M, S) \cong S \otimes_A A_B$ ($\cong S \downarrow_H$). Set $S_B = S \otimes_A A_B$.

**Proposition 3.3.1.**

(a) *For any $\lambda \in I$, we have* $\mathrm{top}(T_\lambda \otimes_B A) \cong S_\lambda \cong \mathrm{soc}(T_{\widetilde{\lambda}} \otimes_B A)$ *and* $\mathrm{soc}(S_{\lambda B}) \cong T_\lambda \cong \mathrm{top}(S_{\widetilde{\lambda} B})$.
(b) *Let $\mu \in K_u$ ($u \neq -1$). If $S_\lambda$ ($\lambda \in I_t$) is a composition factor of* $\mathrm{heart}(T_\mu \otimes_B A)$*, then $u < t$.*
(c) *Let $\lambda \in J_t$ ($t \neq -1$). If $T_\mu$ ($\mu \in K_u$) is a composition factor of* $\mathrm{heart}(S_{\lambda B})$*, then $u < t$.*

**Proof.** (a) follows from 3.2(1) and (3), (b) follows from Lemma 3.3.2(a), and (c) follows from Lemma 3.3.2(b). $\square$

## 4. Proof and remark

Throughout this section, we keep all the notations from 3.3.

*4.1.* We define the ordered algebras $A^0, A^1, \ldots, A^s, A^{s+1}$ by $A^0 = A$ and $A^{t+1} = \mathrm{End}_{K^b(\text{mod-}A^t)}(A^t(I_t)^\bullet)$ for $t = 0, 1, \ldots, s$.

**Proposition 4.1.1.** *Let $t$ be any integer with $0 \leqslant t \leqslant s$.*

(a) *$A^t$ is derived equivalent to $A$.*
(b) *There exists a (unitary) $k$-algebra monomorphism from $B$ to $A^t$ and we have ${}_BA^t_B \cong {}_BB_B \oplus$ (proj.$(B, B)$-bimodule), hence ${}_BA^t_{A^t}$ induces a stable equivalence of Morita type between $A^t$ and $B$. Moreover, ${}_BA^t_{A^t}$ has no nonzero projective summands.*
(c) *${}_A(A \otimes_B A^t)_{A^t}$ is isomorphic to a direct sum of a nonprojective indecomposable module (denoted by $L^t$) and a projective module.*

(d) *Set* $S^t_\lambda = T_\lambda \otimes_B A^t$ *if* $\lambda \in K_{\leqslant t-1}$, *and* $S^t_\lambda = S_\lambda \otimes_A L^t$ *if* $\lambda \in K_{\geqslant t}$. *Then* $S_\lambda$ $(\lambda \in I)$ *are all nonisomorphic simple* $A^t$*-modules.*
(e) *If* $\lambda \in J_{\leqslant t-1}$, *then every composition factor of* $S_\lambda \otimes_A L^t$ *is isomorphic to* $S^t_\mu$ *for some* $\mu \in K_{\leqslant t-1}$.
(f) $K_t$ *satisfies Condition* 2.3.1.

**Proof.** Use induction on $t$. To begin with, suppose $t = 0$.

(a) This is clear since $A^0 = A$.

(b) Let $A_0$, $A_1(= A)$ and $A_2$ be the principal block, a nonprincipal block with full defect, and a block with defect zero, respectively (so $kG = A_0 \oplus A_1 \oplus A_2$), and let $B_0$ and $B_1(= B)$ be the Brauer correspondent of $A_0$ and $A_1$, respectively (so $kH = B_0 \oplus B_1$). Since the $kH$-modules $A_0 \downarrow_H$ and $A_2 \downarrow_H$ belong to the block $B_0$, we have

$$1_{B_1} = 1_{B_1} 1_{kG} = 1_{B_1} 1_{A_0} + 1_{B_1} 1_{A_1} + 1_{B_1} 1_{A_2} = 1_{B_1} 1_{A_1}.$$

In turn, since the $kH$-module $A_1 \downarrow_H$ belongs to $B_1$, we have

$$1_{A_1} = 1_{kH} 1_{A_1} = 1_{B_0} 1_{A_1} + 1_{B_1} 1_{A_1} = 1_{B_1} 1_{A_1}.$$

So we have $1_{B_1} = 1_{A_1}$ and $B_1 = B_1 \cdot 1_{A_1} \subseteq A_1$. By recalling that ${}_B A_A$ induces a stable equivalence of Morita type between $A$ and $B$, we also have ${}_B A_B \cong B \oplus (\text{proj.}(B, B)\text{-bimodule})$. Moreover, for any $\lambda \in I$, $T_\lambda \otimes_B A = T_\lambda \uparrow^G$ is indecomposable, so ${}_B A_A$ has no nonzero projective summands.

(c) This is clear since $A^0 = A$ and ${}_A A_A$ is indecomposable.

(d) This is clear since $L^0 = {}_A A_A$.

(e) This is clear since $J_{\leqslant -1}$ is empty.

(f) We have to check that Condition 2.3.1 holds.

(1) For any $i \in K_0$, $T_i \otimes_B A(= T_i \uparrow^G)$ is nonsimple and (i) follows from 3.2(3) and (ii) follows from 3.2(1).

(2)(i) Since $J_0$ contains 1, $I_0$ contains $q-2$. Now $P_{q-2}$ is a uniserial module whose Loewy series is

$$\operatorname{rad}^{i-1}(P_{q-2})/\operatorname{rad}^i(P_{q-2}) = \begin{cases} S_{q-2} & \text{if } i = 1 \text{ or } 2n+1, \\ S_{q-2p^{k-1}} & \text{if } i = k \text{ or } 2n+2-k \text{ for } 2 \leqslant k \leqslant n, \\ S_1 & \text{if } i = n+1, \end{cases}$$

see [1]. So it follows that

$$\dim \operatorname{Hom}_A(P_{q-2}, P_{q-2}) = \begin{cases} 2 & \text{if } q-2 \neq \widetilde{q-2}\ (=1), \\ 3 & \text{if } q-2 = \widetilde{q-2}\ (=1). \end{cases}$$

Moreover, for any $l \in I_0$, $P_l$ is obtained from $P_{q-2}$ by Frobenius twist, so $\operatorname{Hom}_A(P_l, P_l)$ has the same dimension as $\operatorname{Hom}_A(P_{q-2}, P_{q-2})$.

Before verifying (ii) and (iii) for arbitrary $n$, we shall consider the case $n = 1$. In this case, $I_0 = \widetilde{J_0}$ is just the singleton set $\{q-2\}$ and so the conditions clearly hold, so we may assume $n \geqslant 2$.

(ii) We have to show that $\operatorname{Hom}_A(P_i, P_l) = 0$ for any $i \in I_0$ and any $l \in I_0 - \{i, \tilde{i}\}$, but it suffices to prove it only when $i = q-2$ for the same reason as in (i). Now we have to show that $S_{q-2}$ is not a composition factor of heart$(P_l)$, but since $P_l$ is obtained by Frobenius twist from $P_{q-2}$ whose heart has $S_{q-2p^{k-1}}$ with all $k = 2, 3, \ldots, n$ and $S_1$ as nonisomorphic composition factors, it suffices to show that $S_{q-2}$ is not obtained from these composition factors by Frobenius twists. But this is clear since $q-2 \nsim q-2p^{k-1}$ for all $k = 2, 3, \ldots, n$ and $q-2 \nsim 1$.

(iii) We have to show that $\dim \operatorname{Hom}_A(P_i, P_{\tilde{i}}) = 1$ for $i \in I_0$ with $i \neq \tilde{i}$. It suffices to show only in the case of $i = q-2$. But clearly $q-2 \neq 1 = \widetilde{q-2}$ and $\dim \operatorname{Hom}_A(P_{q-2}, P_1) = \dim \operatorname{Hom}_A(P_1, P_{q-2}) = 1$.

(3) For $\mu \in K_u$, if $S_\lambda$ $(\lambda \in I_t)$ is a composition factor of $T_\mu \otimes_B A$, then $u \leqslant t$ by Proposition 3.3.1(a), (b). Therefore, for any $i \in I_0$, $\mathrm{Hom}_A(P_i, T_l \otimes_B A) = 0$ if $l \notin K_0$, so the proof is complete when $t = 0$.

Next, suppose that the proposition holds for $t$ with $t = 0, 1, \ldots, s-1$, and show that it also holds for $t+1$.

(a) By induction, $K_t$ satisfies Condition 2.3.1 as well as $A^t$ is derived equivalent to $A$, so $A^t(I_t)^\bullet$ is a tilting complex for $A^t$ by Proposition 2.3.1. Therefore, $A^{t+1}$ is derived equivalent to $A^t$ by the definition of $A^{t+1}$.

(b) By induction and Proposition 2.3.1(a), $I_t$ satisfies Condition 2.1.1. So we can use all results in 2.2. Now the result follows from Proposition 2.2.1.

(c) This is clear since ${}_A(A \otimes_B A^{t+1})_{A^{t+1}}$ induces a stable equivalence of Morita type between $A$ and $A^{t+1}$.

(d) We shall proceed in steps:

**Step 1.** For any $\lambda \in K_{\leqslant t-1}$, $(T_\lambda \otimes_B A^{t+1})_{A^{t+1}}$ is simple.

**Step 2.** For any $\lambda \in K_t$, $(T_\lambda \otimes_B A^{t+1})_{A^{t+1}}$ is simple.

**Step 3.** For any $\lambda \in K_{\geqslant t+1}$, $(S_\lambda \otimes_A L^{t+1})_{A^{t+1}}$ is simple.

(When $t = 0$, $K_{\leqslant t-1}$ is empty and so Step 1 is not necessary.)

For $\lambda \in K_{\leqslant t-1}$ $(t \geqslant 1)$, we have $\lambda \notin I_t$ and $S_\lambda^t = T_\lambda \otimes_B A^t$ is simple by induction. Then $\lambda$ satisfies Condition 2.2.1(b), so by Proposition 2.2.3(b), $(T_\lambda \otimes_B A^{t+1})_{A^{t+1}}$ is simple, and Step 1 follows.

Now $K_t$ satisfies Condition 2.3.1 by induction, and hence it follows from Proposition 2.3.1(b) that any $i \in I_t$ satisfies Condition 2.2.1(a) and any $j \in K_t - I_t$ satisfies Condition 2.2.1(b). Therefore, for any $\lambda \in K_t$, by Proposition 2.2.3, $(T_\lambda \otimes_B A^{t+1})_{A^{t+1}}$ is simple, and Step 2 follows.

For $\lambda \in K_{\geqslant t+1}$, $S_\lambda^t = S_\lambda \otimes_A L^t$ is simple by induction, and then for any $\mu \in I_t$, we have

$$\begin{aligned}
\mathrm{Hom}_{A^t}\big(T_\mu \otimes_B A^t, S_\lambda^t\big) &= \underline{\mathrm{Hom}}_{A^t}\big(T_\mu \otimes_B A^t, S_\lambda^t\big) \\
&\cong \underline{\mathrm{Hom}}_{A^t}\big(T_\mu \otimes_B A^t, S_\lambda \otimes_A A \otimes_B A^t\big) \\
&\cong \underline{\mathrm{Hom}}_B(T_\mu, S_\lambda \otimes_A A_B) \\
&\cong \underline{\mathrm{Hom}}_A(T_\mu \otimes_B A, S_\lambda) \\
&= 0,
\end{aligned}$$

where $\underline{\mathrm{Hom}}$ denotes a set of morphisms in the stable module category. Similarly, we obtain $\mathrm{Hom}_{A^t}(S_\lambda^t, T_\mu \otimes_B A^t) \cong \underline{\mathrm{Hom}}_A(S_\lambda, T_\mu \otimes_B A) = 0$. Therefore, for each $\lambda \in K_{\geqslant t+1}$, $\mathrm{Hom}_{A^t}(S_\lambda^t, T_\mu \otimes_B A^t) = 0 = \mathrm{Hom}_{A^t}(T_\mu \otimes_B A^t, S_\lambda^t)$, for any $\mu \in I_t$. Now by Proposition 2.2.2, for $\lambda \in K_{\geqslant t+1}$, we have that $(S_\lambda^t \otimes_{A^t} N^{t*})_{A^{t+1}}$, $N^t$ being a nonprojective indecomposable summand of ${}_{A^{t+1}}(A^{t+1} \otimes_B A^t)_{A^t}$, is simple. In turn, since

$$\begin{aligned}
S_\lambda^t \otimes_{A^t} N^{t*} &= \big(S_\lambda \otimes_A L^t\big) \otimes_{A^t} N^{t*} \\
&\cong S_\lambda \otimes_A \big(L^{t+1} \oplus \big(\text{proj.}\big(A, A^{t+1}\big)\text{-bimodule}\big)\big) \\
&\cong S_\lambda \otimes_A L^{t+1} \oplus \big(\text{proj. } A^{t+1}\text{-module}\big),
\end{aligned}$$

it follows from Krull–Schmidt Theorem that $S_\lambda \otimes_A L^{t+1}$ is simple, so Step 3 follows and (d) is proved.

(e) Let $\lambda \in J_{\leqslant t}$. Then $(S_\lambda \otimes_A L^{t+1})_{A^{t+1}}$ is a direct summand of $S_{\lambda B} \otimes_B A^{t+1}$ since ${}_A(A \otimes_B A^{t+1})_{A^{t+1}} \cong {}_A L^{t+1}_{A^{t+1}} \oplus (\text{proj.}(A, A^{t+1})\text{-bimodule})$ by (b). In turn, by Proposition 3.3.1(a), (c), every composition factor of $S_{\lambda B}$ is isomorphic to some $T_\mu$ $(\mu \in K_{\leqslant t})$, whose tensor product $(T_\mu \otimes_B A^{t+1})_{A^{t+1}}$

($= S_\mu^{t+1}$) is simple. Now since ${}_BA^{t+1}$ is flat, it follows that every composition factor of $(S_{\lambda B} \otimes_B A^{t+1})_{A^{t+1}}$ is isomorphic to some $S_\mu^{t+1}$ with $\mu \in K_{\leqslant t}$, (e) is proved.

(f) We have to prove that $K_{t+1}$ satisfies Condition 2.3.1.

(1) (i) For any simple $A$-module $S$ and any simple $B$-module $T$,

$$\begin{aligned}\underline{\mathrm{Hom}}_{A^{t+1}}\big(T \otimes_B A^{t+1}, S \otimes_A L^{t+1}\big) &\cong \underline{\mathrm{Hom}}_{A^{t+1}}\big(T \otimes_B A^{t+1}, S \otimes_A A \otimes_B A^{t+1}\big)\\ &\cong \underline{\mathrm{Hom}}_B(T, S \otimes_A A_B)\\ &\cong \underline{\mathrm{Hom}}_A(T \otimes_B A, S)\\ &= \mathrm{Hom}_A(T \otimes_B A, S). \qquad (*)\end{aligned}$$

Similarly, we obtain

$$\underline{\mathrm{Hom}}_{A^{t+1}}\big(S \otimes_A L^{t+1}, T \otimes_B A^{t+1}\big) \cong \mathrm{Hom}_A(S, T \otimes_B A). \qquad (**)$$

When $l \in K_{t+1}$ and $i \in K_{\geqslant t+1}$, $(T_l \otimes_B A^{t+1})_{A^{t+1}}$ is nonprojective and indecomposable and $(S_i \otimes_A L^{t+1})_{A^{t+1}}$ is simple, so

$$\begin{aligned}\mathrm{Hom}_{A^{t+1}}\big(T_l \otimes_B A^{t+1}, S_i \otimes_A L^{t+1}\big) &= \underline{\mathrm{Hom}}_{A^{t+1}}\big(T_l \otimes_B A^{t+1}, S_i \otimes_A L^{t+1}\big)\\ &\cong \mathrm{Hom}_A(T_l \otimes_B A, S_i) \quad \big(\text{by } (*)\big)\\ &\cong \begin{cases} 0 & \text{if } i \neq l,\\ k & \text{if } i = l.\end{cases}\end{aligned}$$

On the other hand, when $l \in K_{t+1}$ and $j \in K_{\leqslant t}$, $T_j \otimes_B A^{t+1}$ is simple and so

$$\begin{aligned}\mathrm{Hom}_{A^{t+1}}\big(T_l \otimes_B A^{t+1}, T_j \otimes_B A^{t+1}\big) &= \underline{\mathrm{Hom}}_{A^{t+1}}\big(T_l \otimes_B A^{t+1}, T_j \otimes_B A^{t+1}\big)\\ &\cong \underline{\mathrm{Hom}}_B(T_l, T_j)\\ &= \mathrm{Hom}_B(T_l, T_j)\\ &= 0.\end{aligned}$$

Now it follows that if $l \in K_{t+1}$, then $\mathrm{top}(T_l \otimes_B A^{t+1}) \cong S_l^{t+1}$. Similarly, we can obtain $\mathrm{soc}(T_{\tilde{l}} \otimes_B A^{t+1}) \cong S_l^{t+1}$ by using $(**)$, and (i) follows.

Now we have to verify that $T_i \otimes_B A^{t+1}$ is nonsimple for each $i \in K_{t+1}$. If $i \neq \tilde{i}$, this is satisfied automatically from (i), so suppose $i = \tilde{i}$. Consider the commutative diagram

$$\begin{array}{ccc}\underline{\mathrm{Hom}}_{A^{t+1}}(T_i \otimes_B A^{t+1}, S_i^{t+1}) \otimes \underline{\mathrm{Hom}}_{A^{t+1}}(S_{\tilde{i}}^{t+1}, T_i \otimes_B A^{t+1}) & \xrightarrow{\;m\;} & \underline{\mathrm{Hom}}_{A^{t+1}}(S_{\tilde{i}}^{t+1}, S_i^{t+1})\\ \uparrow & & \uparrow\\ \underline{\mathrm{Hom}}_A(T_i \otimes_B A, S_i) \otimes \underline{\mathrm{Hom}}_A(S_{\tilde{i}}, T_i \otimes_B A) & \xrightarrow{\;m'\;} & \underline{\mathrm{Hom}}_A(S_{\tilde{i}}, S_i)\end{array}$$

where the vertical arrows denote natural isomorphisms, and $m$, $m'$ denote compositions of maps. Now we can replace $\underline{\mathrm{Hom}}$ by Hom in the diagram and $T_i \otimes_B A$ is nonsimple, hence $m'$ must be the zero map. Therefore, $m$ also must be the zero map and $T_i \otimes_B A^{t+1}$ is nonsimple, and hence (i) is proved.

(ii) For $i \in K_{t+1}$ and $l \in I$, we have $S_i^{t+1} = S_i \otimes_A L^{t+1}$ and

$$\begin{aligned}
\mathrm{Hom}_B\bigl(T_l, \mathrm{Hom}_{A^{t+1}}\bigl({}_B A^{t+1}_{A^{t+1}}, S_i^{t+1}\bigr)\bigr) &\cong \mathrm{Hom}_B\bigl(T_l, S_i^{t+1} \otimes_{A^{t+1}} A^{t+1}_B\bigr) \\
&\cong \mathrm{Hom}_{A^{t+1}}\bigl(T_l \otimes_B A^{t+1}, S_i^{t+1}\bigr) \\
&= \underline{\mathrm{Hom}}_{A^{t+1}}\bigl(T_l \otimes_B A^{t+1}, S_i^{t+1}\bigr) \\
&\cong \mathrm{Hom}_A(T_l \otimes_B A, S_i) \quad \bigl(\text{by } (*)\bigr) \\
&\cong \begin{cases} 0 & \text{if } l \neq i, \\ k & \text{if } l = i, \end{cases}
\end{aligned}$$

and hence it follows that $\mathrm{soc}(\mathrm{Hom}_{A^{t+1}}({}_B A^{t+1}_{A^{t+1}}, S_i^{t+1})) \cong T_i$. Similarly, using $(**)$, we have

$$\begin{aligned}
\mathrm{Hom}_B\bigl(\mathrm{Hom}_{A^{t+1}}\bigl({}_B A^{t+1}_{A^{t+1}}, S_{\widetilde{i}}^{t+1}\bigr), T_l\bigr) &\cong \mathrm{Hom}_A(S_{\widetilde{i}}, T_l \otimes_B A) \\
&\cong \begin{cases} 0 & \text{if } l \neq i, \\ k & \text{if } l = i, \end{cases}
\end{aligned}$$

and hence it follows that $\mathrm{top}(\mathrm{Hom}_{A^{t+1}}({}_B A^{t+1}_{A^{t+1}}, S_{\widetilde{i}}^{t+1})) \cong T_i$, and (ii) follows.

Now we shall prove the following lemma:

**Lemma 4.1.1.** *For $\mu \in K_{\geqslant t+1}$ and $\nu \in I_{t+1}$, if $S_\nu^{t+1}$ is a composition factor of $T_\mu \otimes_B A^{t+1}$, then $\mu = \nu$ or $\widetilde{\nu}$.*

**Proof.** Now $T_\mu \otimes_B A^{t+1}$ is a direct summand of $(T_\mu \otimes_B A) \otimes_A L^{t+1}$ since $T_\mu \otimes_B A^{t+1}$ is indecomposable, and every composition factor of $\mathrm{heart}(T_\mu \otimes_B A)$ is isomorphic to some $S_\rho$ with $\rho \in I_{\geqslant t+2} \cup J_{\leqslant t} \cup J_{\geqslant t+1}$ by Proposition 3.3.1(b). So there are three cases:

**Case 1.** If $\rho \in J_{\leqslant t}$, then by (e), every composition factor of $S_\rho \otimes_A L^{t+1}$ is isomorphic to some $S_{\rho'}^{t+1}$ with $\rho' \in K_{\leqslant t}$.

**Case 2.** If $\rho \in J_{\geqslant t+1}$, then by (d), $S_\rho \otimes_A L^{t+1} = S_\rho^{t+1}$.

**Case 3.** If $\rho \in I_{\geqslant t+2}$, then by (d), $S_\rho \otimes_A L^{t+1} = S_\rho^{t+1}$.

Now Case 2 is the only case that $S_\nu^{t+1}$ may be a composition factor of $S_\rho \otimes_A L^{t+1}$. If so, then $S_\nu^{t+1} \cong S_\rho^{t+1}$, hence $S_\nu \cong S_\rho$ and $\rho = \nu \in I_{t+1}$, which is contradicted to Proposition 3.3.1(b). So it follows that $S_\mu^{t+1}$ can be only in the top or socle of $(T_\mu \otimes_B A) \otimes_A L^{t+1}$. Therefore, $S_\mu^{t+1}$ can be only in the top or socle of $T_\mu \otimes_B A^{t+1}$, and hence $\mu$ must be $\nu$ or $\widetilde{\nu}$. $\square$

Before proving (2), we shall show (3).

(3) Let $P_i^{t+1}$ be the projective cover of $S_i^{t+1}$. For $\nu \in I_{t+1}$, we have to show that $\mathrm{Hom}_{A^{t+1}}(P_\nu^{t+1}, T_\mu \otimes_B A^{t+1}) = 0$ for all $\mu \in I - K_{t+1}$. But if $\mu \in K_{\leqslant t}$, this follows since $T_\mu \otimes_B A^{t+1} = S_\mu^{t+1}$, and if $\mu \in K_{\geqslant t+2}$, this follows from the previous lemma.

To prove (2), we shall make some preparation. Let $\mu \in K_{t+1}$. Then we can consider the following three cases:

**Case 4.** If $\mu \neq \widetilde{\mu}$ and $\mu \in I_{t+1}$, then it follows from the previous lemma that $\mathrm{top}(T_\mu \otimes_B A^{t+1})$ $(\cong S_\mu^{t+1})$ is the only composition factor of $T_\mu \otimes_B A^{t+1}$ with the form $S_\nu^{t+1}$, $\nu \in I_{t+1}$.

**Case 5.** If $\mu \neq \widetilde{\mu}$ and $\mu \in J_{t+1}$, then it follows from the previous lemma that $\mathrm{soc}(T_\mu \otimes_B A^{t+1})$ ($\cong S_{\widetilde{\mu}}^{t+1}$) is the only composition factor of $T_\mu \otimes_B A^{t+1}$ with the form $S_\nu^{t+1}$, $\nu \in I_{t+1}$.

**Case 6.** If $\mu = \widetilde{\mu}$, then it follows from the previous lemma that $\mathrm{top}(T_\mu \otimes_B A^{t+1})$ and $\mathrm{soc}(T_\mu \otimes_B A^{t+1})$, which are isomorphic to $S_\mu^{t+1}$, are the only composition factors of $T_\mu \otimes_B A^{t+1}$ with the form $S_\nu^{t+1}$, $\nu \in I_{t+1}$.

**Lemma 4.1.2.** *If $\lambda \in K_{t+1}$, then $P_{\widetilde{\lambda}}^{t+1}$ is a direct summand of $S_{\lambda B} \otimes_B A^{t+1}$.*

**Proof.** Since ${}_A(A \otimes_B A^{t+1})_{A^{t+1}} \cong L^{t+1} \oplus (\text{proj.}(A, A^{t+1})\text{-module})$, we can write $S_{\lambda B} \otimes_B A^{t+1} \cong S_\lambda^{t+1} \oplus R_\lambda$, where $R_\lambda$ is a projective $A^{t+1}$-module. Now $\mathrm{top}(S_{\lambda B}) \cong T_{\widetilde{\lambda}}$, so $T_{\widetilde{\lambda}} \otimes_B A^{t+1}$ is a nonsimple homomorphic image of $S_{\lambda B} \otimes_B A^{t+1}$ with $\mathrm{top}(T_{\widetilde{\lambda}} \otimes_B A^{t+1}) \cong S_{\widetilde{\lambda}}^{t+1}$. Therefore, $S_{\widetilde{\lambda}}^{t+1}$ must appear in $\mathrm{top}(R_\lambda)$. □

Now we shall look at the composition factors of $S_{\lambda B} \otimes_B A^{t+1}$ ($\lambda \in J_{t+1}$). Since the left $B$-module ${}_B A^{t+1}$ is flat, the $A^{t+1}$-module $S_{\lambda B} \otimes_B A^{t+1}$ has the filtration of the form

$$\left( \begin{array}{c} T_{\widetilde{\lambda}} \otimes_B A^{t+1} \\ \hline (\mathrm{heart}(S_{\lambda B})) \otimes_B A^{t+1} \\ \hline T_\lambda \otimes_B A^{t+1} \end{array} \right).$$

So we shall look at $(\mathrm{heart}(S_{\lambda B})) \otimes_B A^{t+1}$:

**Fact 1.** *For any $\lambda \in J_{t+1}$, every composition factor of $(\mathrm{heart}(S_{\lambda B})) \otimes_B A^{t+1}$ is isomorphic to some $T_\mu \otimes_B A^{t+1} = S_\mu^{t+1}$ ($\mu \in K_{\leqslant t}$). In particular, $(\mathrm{heart}(S_{\lambda B})) \otimes_B A^{t+1}$ has no composition factors with the form $S_i^{t+1}$, $i \in I_{t+1}$.*

**Proof.** By Proposition 3.3.1(c), every composition factor of $\mathrm{heart}(S_{\lambda B})$ is isomorphic to some $T_\mu$ ($\mu \in K_{\leqslant t}$), whose tensor product $T_\mu \otimes_B A^{t+1}$ is simple. □

(2)(i) Let $\lambda \in J_{t+1}$.

- Assume $\lambda \neq \widetilde{\lambda}$.
  By Cases 4, 5 and Fact 1, we see that $S_{\widetilde{\lambda}}^{t+1}$ appears in $S_{\lambda B} \otimes_B A^{t+1}$ as a composition factor with multiplicity 2. Now $S_\lambda^{t+1} \oplus P_{\widetilde{\lambda}}^{t+1}$ is a direct summand of $S_{\lambda B} \otimes_B A^{t+1}$ and $\mathrm{top}(P_{\widetilde{\lambda}}^{t+1}) \cong \mathrm{soc}(P_{\widetilde{\lambda}}^{t+1}) \cong S_{\widetilde{\lambda}}^{t+1}$, so $S_{\widetilde{\lambda}}^{t+1}$ appears in $S_\lambda^{t+1} \oplus P_{\widetilde{\lambda}}^{t+1}$ as a composition factor with multiplicity 2, as desired.
- Assume $\lambda = \widetilde{\lambda}$.
  By Case 6 and Fact 1, we see that $S_{\widetilde{\lambda}}^{t+1}$ appears in $S_{\lambda B} \otimes_B A^{t+1}$ as a composition factor with multiplicity 4, hence appears in $P_\lambda^{t+1}$ with multiplicity 3 or less. Now $T_{\widetilde{\lambda}} \otimes_B A^{t+1}$ is a proper quotient of $P_{\widetilde{\lambda}}^{t+1}$ and $\mathrm{top}(T_{\widetilde{\lambda}} \otimes_B A^{t+1}) \cong \mathrm{soc}(T_{\widetilde{\lambda}} \otimes_B A^{t+1}) \cong S_\lambda^{t+1}$ ($\cong S_{\widetilde{\lambda}}^{t+1}$), so $S_{\widetilde{\lambda}}^{t+1}$ appears in $P_{\widetilde{\lambda}}^{t+1}$ with multiplicity just 3, as desired.

(ii) Let $\lambda, \mu \in J_{t+1}$ with $\widetilde{\lambda} \neq \widetilde{\mu}$ and $\widetilde{\lambda} \neq \mu$. Since $P_{\widetilde{\mu}}^{t+1}$ is a direct summand of $S_{\mu B} \otimes_B A^{t+1}$ by Lemma 4.1.2, it suffices to show that $\mathrm{Hom}_{A^{t+1}}(P_{\widetilde{\lambda}}^{t+1}, S_{\mu B} \otimes_B A^{t+1}) = 0$. By Cases 4–6 and Fact 1, it follows that $S_{\widetilde{\mu}}^{t+1}$ is the only composition factor of $S_{\mu B} \otimes_B A^{t+1}$ whose form is $S_\nu^{t+1}$, $\nu \in I_{t+1}$. But now $\widetilde{\lambda} \neq \widetilde{\mu}$ and $\widetilde{\lambda} \neq \mu$, so $S_{\widetilde{\lambda}}^{t+1}$ does not appear in $S_{\mu B} \otimes_B A^{t+1}$ as a composition factor and (ii) follows.

(iii) It is well known that if two (finite dimensional) $k$-algebras $R_1$ and $R_2$ are derived equivalent, then those Cartan matrices $C_{R_1}$ and $C_{R_2}$ satisfy $C_{R_1} = {}^tPC_{R_2}P$ for some $P \in GL(l, \mathbb{Z})$, where ${}^tP$ is the transpose of $P$ and $l$ is the size of the matrices $C_{R_1}$ and $C_{R_2}$ (cf. [3, 4.2. Proposition]). This implies that the positive definiteness of a Cartan matrix is preserved under derived equivalence. Since Cartan matrices of group algebras of finite groups are positive definite, it follows that $A^{t+1}$ is also positive definite. So this condition automatically holds and the proof is complete. □

To prove the main theorem, we have to show that $A^{s+1}$ is derived equivalent to $B$. But in fact, it is proved that they are isomorphic as $k$-algebras:

**Proposition 4.1.2.** *$A^{s+1}$ is isomorphic to $B$ as $k$-algebras.*

**Proof.** For any $\mu \in K_{\leqslant s-1}$, by Proposition 4.1.1(d), $T_\mu \otimes_B A^s$ is simple, so $\mu$ satisfies Condition 2.2.1(b), namely, $\dim \mathrm{Hom}_A(T_\mu \otimes_B A^s, T_l \otimes_B A^s) = \delta_{\mu l}$ for any $l \notin I_s$. On the other hand, by Proposition 4.1.1(f), $K_s$ satisfies Condition 2.3.1, so by Proposition 2.3.1, any $i \in I_s$ and any $j \in K_s - I_s$ satisfies Condition 2.2.1(a) and (b), respectively. Now it follows that $I_s$ satisfies Condition 2.2.1, namely, any $i \in I_s$ and any $j \in I - I_s$ satisfy Condition 2.2.1(a) and (b) respectively, so by Corollary 2.2.1, $A^s(I_s)^\bullet$ is a Rickard tilting complex for $(B, A^s)$. Therefore, we have

$$\mathrm{Hom}^\bullet_{A^s}\big({}_BA^{s\bullet}_{A^s}, {}_BA^{s\bullet}_{A^s}\big) \cong {}_BA^{s\bullet} \overset{\bullet}{\otimes}_{A^s} A^{s\bullet *}_B \cong {}_BB_B \quad \big(\text{in } K^b\big(\text{mod-}B^{op} \otimes B\big)\big),$$

where $A^{s\bullet}$ denotes $A^s(I_s)^\bullet$. By taking the cohomology at degree 0, we have ${}_BA^{s+1}_B \cong {}_BB_B$ in $K^b(\text{mod-}B^{op} \otimes B)$, hence ${}_BA^{s+1}_B \cong {}_BB_B$ as $(B, B)$-bimodules. But now there is an algebra monomorphism from $B$ to $A^{s+1}$, so $B$ must be isomorphic to $A^{s+1}$ as $k$-algebras. □

**Proof of Theorem 1.1.1.** By the proof of Proposition 4.1.2, $A^s(I_s)^\bullet$ is a tilting complex for $A^s$, and so $A^{s+1}$ is derived equivalent to $A^s$. Now the theorem follows from Propositions 4.1.1(a) and 4.1.2. □

*4.2.*

**Remark.** Marcus's papers [8, 3.4. Theorem] and [9, 3.13. Proposition] imply that Broué's abelian defect group conjecture holds for the blocks of $kGL(2, q)$ covering the principal block of $kSL(2, q)$ (see [9, 3.14. Example]). But the conjecture also holds for those covering the nonprincipal block of $kSL(2, q)$ by imitating the argument in those papers.

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